Pergamon

J. Appl. Maths Mechs, Vol. 62, No. 2, pp. 319–321, 1998

PII: S0021-8928(98)00042-2
0021–8928/98/$—see front matter

THE RELATION BETWEEN THE EQUATIONS OF THE TWO-DIMENSIONAL THEORY OF ELASTICITY FOR ANISOTROPIC AND ISOTROPIC BODIES†

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(*Received* 2 *April* 1997)

The two-dimensional problem of the theory of elasticity for an isotropic body is reduced to the solution of the problem for an anisotropic body. Small additional terms are introduced into the biharmonic operator of the problem of the theory of elasticity of an isotropic body, so that the generalized biharmonic operator obtained has no multiple roots. The general solution is then represented in terms of a function of the generalized complex variables, and numerical investigations for isotropic and anisotropic bodies are carried out using the same algorithms. The effectiveness of such a replacement is demonstrated in numerical investigations for simplify connected and multiply connected regions of arbitrary section. © 1998 Elsevier Science Ltd. All rights reserved.

The solution of the two-dimensional problem of the theory of elasticity for an anisotropic body can be reduced to the integration of the generalized biharmonic equation [1]

$$\left[a_0\frac{\partial^4}{\partial x^4}+a_1\frac{\partial^4}{\partial x^3\partial y}+a_2\frac{\partial^4}{\partial x^2\partial y^2}+a_3\frac{\partial^4}{\partial x\partial y^3}+a_4\frac{\partial^4}{\partial y^4}\right]F=0 \tag{1}$$

where $a_0, \ldots, a_4$ are real constants, which depend on the elastic properties of the body considered.

The solution of this equation is sought in the form $F(x, y) = F(z)$, $z = x + \mu y$ and depends on the roots of the corresponding characteristic equation.

These roots are either complex or pure imaginary pairwise conjugate

$$\mu_j=\alpha_j+i\beta_j,\quad \bar{\mu}_j=\alpha_j-i\beta_j,\quad j=1,2 \tag{2}$$

After finding them, Eq. (1) can be written in the form

$$\Delta_1\Delta_2F=0,\quad \Delta_j=\mu_j\bar{\mu}_j\frac{\partial^2}{\partial x^2}-(\mu_j+\bar{\mu}_j)\frac{\partial^2}{\partial x\partial y}+\frac{\partial^2}{\partial y^2} \tag{3}$$

If the roots μ_j are different, the general solution of Eq. (3) is the function

$$F=\varphi_1+\varphi_2 \tag{4}$$

where φ_j satisfy the equations

$$\Delta_j\varphi_j=0,\quad j=1,2 \tag{5}$$

The general real solution of Eq. (5) will be the expression

$$\varphi_j=F_j(z_j)+\overline{F_j(z_j)} \tag{6}$$

Here $F_j(z_j)$ are arbitrary analytic functions of the generalized complex variables $z_j = x + \mu_j y$.
Function (4) takes the form

$$F=2\,\mathrm{Re}[F_1(z_1)+F_2(z_2)] \tag{7}$$

In the case of an isotropic body the characteristic equation has double roots i and $-i$.
The operators of Eq. (3) in this case take the form

†*Prikl. Mat. Mekh.* Vol. 62, No. 2, pp. 344–346, 1998.

$$\Delta_j = \partial^2/\partial x^2 + \partial^2/\partial y^2 \tag{8}$$

and the general real solution can be represented in the form [2]

$$F = 2\,\mathrm{Re}[\bar{z}\varphi(z) + \chi(z)] \tag{9}$$

The solution can be written in the same form in the case of pairwise equal complex roots $\mu_1 = \mu_2 = \alpha + i\beta$. Carrying out the affine transformation $x_* = x + \alpha y, y_* = \beta y$ we obtain the complex variable $z = z_1 = z_2 = x_* + iy_*$. This version can be included in our further consideration as a case which is a complete analogy with the case of an isotropic body.

Solution (9) differs in form from the general solution (7), which corresponds to the case when the roots μ_j are different. This prevents us from carrying out numerical investigations of problems for isotropic and anisotropic bodies using the same algorithms.

Here we propose to reduce the solution of the problem for an isotropic body to the problem for an anisotropic body.

We introduce additional terms into the operators (8) so that they take the form

$$\Delta_1 = (1+\varepsilon)^2\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \Delta_2 = (1-\varepsilon)^2\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

where ε is a small parameter. Then the biharmonic operator is converted into a generalized biharmonic operator. The roots of the characteristic equation μ_j are now multiple and have the form

$$\mu_1 = (1+\varepsilon)i, \quad \mu_2 = (1-\varepsilon)i, \quad \bar{\mu}_1 = -(1+\varepsilon)i, \quad \bar{\mu}_2 = -(1-\varepsilon)i \tag{10}$$

while the function F is written in the form (7).

If we allow ε to tend to zero, the combination of solutions of the form (7) gives a solution of the biharmonic equation of the form (9). Expanding the functions $F_j(z_j) = (-1)^{j+1}F(z_j) + i\varepsilon\Psi_j(z_j)$, where $z_1 = z + i\varepsilon y$, $z_2 = z - i\varepsilon y$, $z = x + iy$, in series in the parameter ε we obtain

$$F_j(z_j) = (-1)^{j+1}[F(z) + (-1)^{j+1}i\varepsilon yF'(z) + \ldots] + i\varepsilon[\Psi_j(z) + (-1)^{j+1}i\varepsilon y\Psi_j'(z) + \ldots], \quad j = 1,2$$

Hence we obtain

$$\lim_{\varepsilon\to 0}[F_1(z_1) - F_2(z_2)] = F(z) + F(z) \tag{11}$$

$$\lim_{\varepsilon\to 0}\left[\frac{F_1(z_1) + F_2(z_2)}{i\varepsilon}\right] = y[F'(z) + F'(z)] + \Psi_1(z) + \Psi_2(z)$$

The first relation of (11) gives the harmonic component of the solution (9), while the second term gives the biharmonic and harmonic components. Adding to them similar combinations for the functions $\overline{F_1}(\overline{z_1})$ and $\overline{F_2}(\overline{z_2})$ we obtain a solution of the form (9). Hence, solution (7) with parameters (10) contains the biharmonic solution (9), which can be isolated by the above method as $\varepsilon \to 0$. This means that for small values of ε the solution (7) can be used to determine the stress-strain state of isotropic media.

In the case of a generalized plane stressed state of a plate, the strains and stresses in it can be found using well-known formulae [1] in terms of the function $\Phi_j(z_j)$ and the parameters p_j and q_j ($j = 1, 2$), where

$$\Phi_j(z_j) = dF_j/dz_j, \quad p_j = a_{11}\mu_j^2 + a_{12}, \quad q_j = a_{12}\mu_j + a_{22}/\mu_j$$

and a_{11}, a_{22} and a_{12} are deformation coefficients of the isotropic plate.

For specified external forces X_n and Y_n, the boundary conditions for determining the functions $\Phi_j(z_j)$ take the form

$$2\,\mathrm{Re}[\Phi_1(z_1) + \Phi_2(z_2)] = -\int_0^s Y_n ds + c_1$$

$$2\,\mathrm{Re}[\mu_1\Phi_1(z_1) + \mu_2\Phi_2(z_2)] = \int_0^s X_n ds + c_2$$

If the displacements u^* and υ^* are specified, they are found as follows:

$$2\,\mathrm{Re}[p_1\Phi_1(z_1) + p_2\Phi_2(z_2)] = u^* + \omega y - u_0$$

$$2\,\mathrm{Re}[q_1\Phi_1(z_1) + q_2\Phi_2(z_2)] = v^* - \omega x - v_0$$

As an example, consider a plate with an elliptic opening. The plate is stretched by forces of intensity p along the x axis. In this case, in a continuous plate

$$\sigma_x^0 = p, \quad \sigma_y^0 = \tau_{xy}^0 = 0 \tag{12}$$

The solution, taking into account the effect of the elliptic opening in the plate, is obtained using the functions [3]

$$\Phi_j(z_j) = \frac{a_j}{\zeta_j}, \quad a_1 = -a_2 = \frac{pb}{2i(\mu_1 - \mu_2)} \tag{13}$$

The quantity ζ_j is related to z_j by the equation

$$z_j = \frac{a - i\mu_j b}{2}\zeta_j + \frac{a + i\mu_j b}{2}\zeta_j^{-1}$$

(a and b are the semiaxes of the ellipse).

The stresses close to the contour of the opening can be calculated from the formula

$$\sigma_\theta^\varepsilon = \sigma_x^0 + \sigma_y^0 + 2\,\mathrm{Re}\sum_{j=1}^{2}(1+\mu_j^2)\Phi_j'(z_j) \tag{14}$$

Substituting (13) into (14) we obtain, after some reduction,

$$\sigma_\theta^\varepsilon = p\left[1 - 2\,\mathrm{Re}\, b\frac{2[(a+b)\sigma^2 - (a-b)] - \varepsilon^2 b(\sigma^2+1)}{[(a+b)\sigma^2 - (a-b)]^2 - \varepsilon^2 b^2(\sigma^2+1)^2}\right] \tag{15}$$

Here $\sigma = \cos\theta + i\sin\theta$ and θ is the polar angle.

The solution of this problem, obtained using the theory of elasticity of an isotropic body, has the form [2]

$$\sigma_\theta = p\left[1 - \mathrm{Re}\frac{4b}{(a+b)\sigma^2 - (a-b)}\right] \tag{16}$$

Comparing formulae (15) and (16) we see that they are identical when $\varepsilon = 0$.

Carrying out the calculations, it can be shown that

$$|\sigma_\theta^\varepsilon - \sigma_\theta| / p \leqslant 2b\varepsilon^2 \tag{17}$$

Hence, the method of replacing the biharmonic equation by the generalized biharmonic equation indicated, enables an approximation to be constructed with an accuracy of the order of ε^2.

In conclusion, we note that similar results also hold for the equations of the bending of thin plates.

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Translated by R.C.G.